Algebra-I
End-Semestral Exam
B. Math - First year
2014-2015

Time: 3 hrs
Max score: 100

Answer all questions.

(1) (a) Prove that if G is a group and A, B subgroups of G such that $A \subseteq N_G(B)$ then $A \cap B$ is a normal subgroup of A and $AB/B \cong A/A \cap B$.
(b) Deduce that if N is a normal subgroup of G of prime index p, and if H is any subgroup of G, either
(i) $H \subseteq N$ or
(ii) $G = NH$ and $|H : H \cap N| = p$. (6+8)

(2) (a) State and prove Cayley's theorem.
(b) Prove that if G is a finite group of order n and p is the smallest prime dividing n, then any subgroup (if it exists) of index p is normal in G. (6+8)

(3) (a) Let G be a group acting on a finite set X. Let $Orb(x)$ denote the orbit of $x \in X$ and let G_x denote the stabiliser of x in G. Show that

$$|Orb(x)| = |G : G_x|.$$

(b) Deduce that if G is a finite group, $g \in G$ and X^g is the set of elements in X which are fixed by g, then the number of orbits of this action is $(\sum_{g \in G} |X^g|)/|G|$. (6+8)

(4) (a) State and prove the **Class Equation**.
(b) Let G be a non-abelian group and let p be a prime divisor of the order of G. Prove that
(i) There exists an element of order p in G.
(ii) If $o(G) = p^n$, $n \geq 2$, centre of the group $Z(G) \neq \{e\}$. (6+6+6)

(5) (a) State Sylow's theorems.
(b) Prove that no group of order 224 is simple. (6+8)

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(6) Compute the number of Sylow subgroups in
(a) A_5, and
(b) S_5. (8+8)

(7) (a) Define semi-direct product of two groups H and K.
(b) Show that the dihedral group D_{2n} of order $2n$ is a semi direct product of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_2$. (4+6)